# Oligopoly and game theory in Vittorio Cafagna's studies

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**Abstract.**
In this paper, I illustrate an extension of Smale's analysis of the repeated prisoner's dilemma with imperfect recall to a dynamic Cournot duopoly game in which firms have bounded memory and rationality and can thus observe only summary statistics of past history. A stable long-run cooperation is achieved by employing a dynamical system that takes into account repeated interactions between firms and a set of behavioural rules, irrespective of the initial conditions.

**Sunto.**
*Oligopolio e teoria dei giochi negli studi di Vittorio Cafagna.* In questo lavoro viene presentata una estensione del modello di Smale sul dilemma del prigioniero ripetuto con memoria imperfetta a un gioco dinamico di duopolio alla Cournot in cui le imprese hanno limiti sostanziali nel custodire ed organizzare le informazioni a disposizione, e possono dunque conservare soltanto un qualche tipo di riassunto dei precedenti accadimenti (una sorta di "valore medio"). Viene dimostrato che è possibile raggiungere un equilibrio cooperativo stabile di lungo periodo attraverso l'implementazione di un sistema dinamico che tiene conto delle interazioni ripetute fra le imprese e di un set di regole di comportamento, e che inoltre prescinde dalle condizioni iniziali.

## 1 Our collaboration: how it all began

Vittorio Cafagna was a family friend, to whom my father was very attached. The first time I talked to him was after my first degree. I was looking for information about

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foreign masters and Ph.D.'s, and he gave me a lot of instructions and suggestions. The impression he made on me in that occasion never changed in the following years, when I had the possibility to know him very well and share with him many hours of study and conversation: Vittorio was kind, helpful, generous, noble, smiling. He used to be a true friend. He loved life and was eager to explore it in all its facets. Several years passed since then, during which there hadn't been any occasion to discuss with him, also because my field of studies was (or better, seemed to me) quite far from his. Later, I happened to develop a theoretical model on the relationship between advertising and entry barriers and asked him for some advices. It was 1998. We talked a lot, and for the first time I heard about his growing interest towards economics and finance. That was the birth of our collaboration.

At first we met to discuss about my industrial organization model (Coccorese, 2001), and more in general about the application of game theory to economics. Actually, in my paper I recall and apply the well-known taxonomy of business strategies by Fudenberg and Tirole (1984): by using an "animal terminology" (*top dog*, *lean and hungry look*, *puppy dog*, *fat cat*) within a two-stage model, it shows under which conditions an incumbent firm chooses to over-invest or under-invest in order to respond to the threat of entry from a potential rival. The model by Fudenberg and Tirole can be applied to both the entry deterrence and the accommodation case, and is able to clarify the functioning of a great deal of commitment models (concerning investments in capacity, advertising, R&D, etc.), where the payoffs of the final period are influenced by some strategic commitment that one of the players can afford in a previous period.[2]

This topic intrigued Vittorio, who began to study some of the milestone contributions, among which there was the game theory manual by Fudenberg and Tirole (1991). Just when dealing with this book, he sent me a nice message: "*I am absorbed in a preliminar coming and going between your paper and the wicked Fudenberg-Tirole. As you know, unfortunately I am not a game theory expert yet, even if with my current rhythm of studying I hope to soon become like that. I am working out a list of issues, some of which self-destroy as I understand more and more. Before long I will mail you those that will survive*".

In that period Vittorio travelled a lot, especially to France. So, given my and his engagements, we found very little time to organize and clearly identify the game theory readings that we wanted to study together and with regular frequency. Besides, I had started to devote part of my research energies to empirical industrial organization.

[2]Regarding the commitment models, see also Tirole (1988), chapter 8.

One year passed, may be a bit more. In the meantime, we met seldom but we were often in touch. During one of our sporadic meetings of that time, we started talking about the book by Fudenberg and Levine (1998), which had been published recently and explored the role of learning in multi-period games. The authors considered the equilibrium of a game as the long-run outcome of a process in which less than fully rational players fumble for optimality over time. In our study plans, this book competed with that by Weibull (1995), which was specifically focused on evolutionary game theory (a sort of meeting point between evolutionary biology and rationalistic economics). Under Vittorio's point of view, the book by Weibull was more simple, because game theory was more mathematically formalized, but at the same time he thought that its content was more limited. Finally, we opted for the book by Fudenberg and Levine, and commenced to dedicate some of our time reading it (on the other hand, we were both quite busy because of our job, and Vittorio also travelled a lot in United States and France).

On average we met twice or three times a month, and discussed the sections of the book we had previously read on our own. One of the topics concerned the *fictitious play*[3]. Within the learning and evolutionary theory, we have a fictitious play dynamics when, under repeated interactions, each player assumes that the strategies of his opponents are randomly chosen from some unknown stationary distribution, and in each period he selects the best response to the historical frequency of actions of his opponents. In other words, each player does not react to what happened in the very last stage, but takes into account the whole story of the game, which is summarized by some proper indicator. He updates this indicator in each round, then chooses the strategy that allows him the highest expected payoff, given the historical distribution of his opponents' strategies.

This kind of conjecture seemed us rational and realistic, especially we foresaw the potential for an application to oligopolistic markets. When there are few firms in a market, it is very likely that their strategic interactions repeat over time; furthermore, possible adjustments regarding the level of the variables under firms' control (quantity or price), as a reaction to rivals' past behaviour, can not be too drastic; finally, it is plausible that every producer has memory bounds that impede the storing of the whole set of opponents' past choices. All the above led us to regard as unrealistic the tit-for-tat solution (which consists in doing what the rival did in the previous stage, and has been demonstrated to be the optimal behaviour to attain cooperation in repeated games[4]), also because in real world accidental deviations or misunderstandings can happen, with the result of locking players in an endless cycle of mutual defection that they would have unquestionably avoided.

[3] The fictitious play was first proposed by Brown (1951) and Robinson (1951).

[4] For further details, see Axelrod (1984).

At the beginning of 2000, Vittorio went to the United States for a short stay. On his return, he told me he had met José Scheinkman, who he knew since long time.[5] In that occasion, they talked about our research interests, and Scheinkman was quite encouraging. He also told Vittorio about an article by Steve Smale that had appeared in Econometrica in 1980. In his opinion, this paper would have been very interesting to us.

So we got Smale's (1980) paper. The renowned mathematician had introduced bounded rationality in the repeated prisoner's dilemma game by assuming that players were not able to keep the memory of all past interactions. Particularly, he moved from the idea that agents have substantive bounds to the storing and organizing of information, so they only keep some kind of summary (a sort of "average value") of past outcomes in their memory. At each stage, they take a decision according to this summary, and are restricted to strategies where the action depends continuously on it. His analysis was therefore based on a dynamical system, without promises or binding agreements.

This approach was quite fascinating, especially because it gave us the possibility of working on a model characterized by a good degree of realism. Actually, it is logically reasonable (and also analytically convenient) that "history-dependent state variables" should be preferred to "history-dependent strategies": over time (when history matters) the latter group tends to require an extraordinary amount of information at every move, while the former represents a reasonable alternative for players with bounded rationality (Clemhout and Wan, 1989, p. 131; Aumann, 1989, pp. 42-43).

After all, Smale had extended the frames of fictitious play by introducing (in the context of the repeated prisoner's dilemma) a class of general strategy updating rules that fed back with the historical distributions of payoffs.[6]

We decided to work on a dynamic Cournot duopoly model, reformulated à la Smale in order to take memory into consideration. We needed to identify a specific set of behavioural rules that would have been able to warrant the achievement of the cooperative solution, so avoiding any sub-optimal outcome. In our intentions (fruits of long and lively discussions), this set of rules should have been characterized by a tendency to avoid unnecessary conflicts, the presence of punishments for defections and forgiveness

[5] José Scheinkman is currently Professor of Economics at Princeton University. He writes: "I was first introduced to Vittorio in 1985, in Paris, by our common friend Henri Berestycki. Vittorio and I met many times in Paris, New York and at least once in Rome, and I have wonderful memories of conversations that ranged over a wide set of topics, including, naturally, mathematics and economics". I wish to thank him for this recollection.

[6] Smale's analysis has been later extended by Benaim and Hirsch (1997) and Ahmed and Hegazi (1999, 2001).

after punishment, and also by transparency and predictability for all players. Our hope was that, in a dynamic framework where many interactions take place and players are neither stupid nor totally without foresight, cooperation could have emerged as long as participants were able to learn during the sequence of game rounds.

We worked on this topic for more than one year, with up and down mood and luck. Sometimes we felt this was an impossible undertaking, other times we were absolutely sure to walk the right way. At last, we hit the target: we built a duopolistic model where two rational firms with bounded memory were able to achieve a stable cooperative equilibrium simply following a set of behavioural rules, without need of stipulating agreements or retaliating.

# 2 The duopoly model

We start with two firms producing a homogeneous good in a given market. Production decisions are taken at discrete time periods $t = 0, 1, 2, 3, \ldots$ The quantity of output by each firm at time $t$ is denoted by $q_{i,t}$, $i = 1, 2$. The cost of production, $C_{i,t}$, is a linear function of the output:

$$C_{i,t} = c_i q_{i,t} \,. \tag{2.1}$$

The inverse demand function is also linear:

$$p_t = a - bQ_t, \tag{2.2}$$

where $a, b > 0$, $a > c_i$ and $Q_t = \sum_i q_{i,t}$.

At time $t$, the profit of each firm, $\pi_{i,t}$, is equal to:

$$\pi_{i,t} = (a - bQ_t - c_i)\, q_{i,t} \,. \tag{2.3}$$

Given the dynamical nature of our Cournot model, we needed to characterize how firms decide the quantity to be produced in period $t + 1$ as a response of the profit accrued in period $t$.
The standard Cournot conjecture holds that each firm expects the rival to leave its quantity unchanged in the next period, which means that $q_{j,t+1} = q_{j,t}$, $j = 1, 2$. Even if this assumption foreshadows a sort of learning behaviour, it did not appear to us particularly clever: both firms behave as if they expect that tomorrow's play by the opponent will be the same as today [7], and choose their strategies ignoring previous experience.

[7] See Fudenberg and Levine (1998), p. 10. Ahmed and Hegazi (2002) formalize a dynamic model where the rival's expected quantity in period $t + 1$ is a weighted function of its previous quantities.

In order to overcome those limitations and make the decision process closer to reality, we strove to change the quantity game by introducing bounded rationality memory effects. It was here that the model proposed by Smale for the solution of the iterative prisoner's dilemma turned out to be extremely valuable.

We had first to construct a two-dimensional space of possible firms' payoffs (i.e. profits). Here, we faced a non-trivial problem: while in Smale's model players can select only among two strategies (*play easy* and *play tough*) and the interaction between their choices drives straight to the payoff matrix, in our framework firms had to choose the output level (the variable under their control) among a set of possibilities that was continuous and hence theoretically infinite. In addition, we needed to move from a quantity space to a profit space, with profits being generated by the interaction between the individual production choices, which also affected the price level.

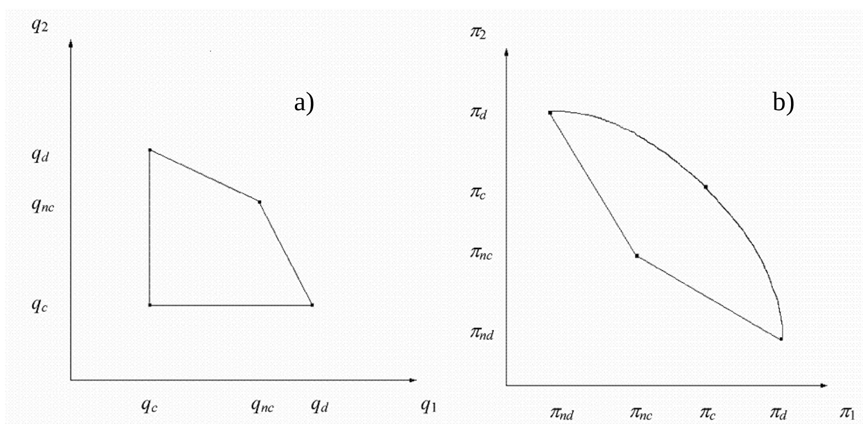



Figure 1: *a) The convex quantity space. b) The conves profit space.*

Figure 1 a) shows the quantity space map. We had also to build the corresponding space map of profits. Given the profit function (2.3), and assuming (for simplicity) that firms have equal size (implying that $c_1 = c_2 = c$), our "non-cooperative" outcome ($nc$) was the standard Cournot-Nash equilibrium, whose quantities and profits for both firms were $q_{nc} = \frac{a-c}{3b}$ and $\pi_{nc} = \frac{(a-c)^2}{9b}$, respectively.

The "cooperative" solution ($c$) coincided with the case of collusion, when firms are supposed to act as a monopolist in the market. Under this conjecture, and assuming that - being equal - they enjoyed the same market share, quantities and profits were $q_c = \frac{a-c}{4b}$ and $\pi_c = \frac{(a-c)^2}{8b}$, respectively, for each of them.

Finally, still in parallel with the prisoner's dilemma, we had to identify the outcome of a unilateral deviation from the cooperative behaviour coming from one of the two firms. In this case, we assumed that the defecting firm maximizes its own profit given the cooperative quantity produced by the other firm. Therefore, the resulting quantities were $q_d = \frac{3(a-c)}{8b}$ and $q_{nd} = q_c = \frac{a-c}{4b}$, and profits were $\pi_d = \frac{9(a-c)^2}{64b}$ and $\pi_{nd} = \frac{3(a-c)^2}{32b}$, where the subscripts $d$ and $nd$ refer to the deviating and the non-deviating firm, respectively.

Note that for our payoffs the following condition holds: $\pi_d - \pi_c < \pi_c - \pi_{nd}$. So, when a firm cooperates and the other defects, the additional reward $\pi_d - \pi_c$ the second firm receives for its deviating behaviour is lower than the penalty $\pi_c - \pi_{nd}$ weighting on the first firm for its cooperating behaviour.[8] This condition also implies that - like the prisoner's dilemma game - in the profit space the quadrilateral with vertices $(\pi_c, \pi_c)$, $(\pi_d, \pi_{nd})$, $(\pi_{nc}, \pi_{nc})$, $(\pi_{nd}, \pi_d)$, representing our space of possible payoffs, is convex. For firm 1, it is depicted in Figure 1 b) (firm 2's state space has an identical graph, but with inverted axes).

In a static framework, the outlined game has a unique Nash equilibrium, where both players choose to behave non-cooperatively (and therefore to defect if the rival tries to cooperate). Actually, this is a dominating strategy for both of them, even though there would be a gain in cooperating, since $\pi_c > \pi_{nc}$. However, considering that in a market the interaction between firms is repeated (and also that price, and hence unit revenue, is a function of the level of production of both firms), there is room for cooperation.

Our purpose was therefore to provide a set of deterministic behaviour rules that allow to reach the cooperative equilibrium $(\pi_c, \pi_c)$, or better to grant firms a long run cumulative profit flow whose average value approaches the cooperation outcome.

The repeated interaction gives rise to a sequence of production levels $\{q_1, q_2, \dots, q_t\}$, yielding the series of profits $\{\pi_1, \pi_2, \dots, \pi_t\}$. Our assumption was that, given players' bounded memory, at time $t$ firm $i$ makes its decision according to its up-to-date average profit, calculated as follows:

$$\pi^*_{i,t} = \frac{\sum_{m=1}^{t} \pi_m}{t}. \tag{2.4}$$

After observing the average profit $\pi^*_{i,t}$ at time $t$, firm $i$ decides about the strategy, which involves the choice of a new level of quantity. In symbols: $q_{i,t+1} = f(\pi^*_{i,t})$. This output, together with the production of the other firm, determines the market price (through the inverse demand function) and the level of profit $\pi_{i,t+1}$ of period $t+1$, which in turn updates the average profit.

[8] If this condition does not hold, in the long run firms could receive more than $\pi_c$ by alternating defection and cooperation (in the sense that when one of them defects, the other cooperates, and viceversa).

It is possible to write the outlined averaging (memory) process as a dynamical system:

$$\pi^*_{i,t+1} = g(\pi^*_{i,t}) = \frac{t\pi^*_{i,t} + \pi^*_{i,t+1}}{t+1}. \tag{2.5}$$

# 3 The 'good' strategy

At this time, we had to establish how firms choose their quantity $q_{i,t+1}$ in period $t$, after having observed $\pi^*_{i,t}$. In our model, we define as firm $i$'s *strategy* the choice of the quantity $q_{i,t} \in S_q$ to be produced (where $S_q$ is the convex quantity space). Every choice $q_{i,t}$ in period $t$ generates a payoff $\pi_{i,t} \in S_\pi$ (with $S_\pi$ representing the convex profit space), that updates the average profit $\pi^*_{i,t}$. The latter constitutes the reference point for the choice of the quantity $q_{i,t+1}$, to be made as follows.
First of all, a *solution* is a pair $(q, \pi)$, where $q$ is a level of output and $\pi$ is the stationary associated payoff, both generated through the dynamics defined by (2.5).
As already noted, while in the prisoner's dilemma each player can choose between two dichotomic strategies ("cooperate" and "defect", let's say), production decisions imply that there exist infinite possibilities of quantity choices. So it is reasonable to imagine that firms approximate both the cooperative and the non-cooperative quantity in a gradual fashion, also considering technological constraints that may discourage sudden production changes.
In our model, we therefore assumed that firms increase (decrease) their output according to a given fraction $\alpha_i$ ($\beta_i$) of the gap between the production of the previous period and the production characterizing the equilibrium they wish to reach.
Let us consider firm 1. First of all, it knows that, having produced $q_{nc}$ in all periods $(1, \ldots, t)$, at time $t$ it has to be $\pi^*_{1,t} \geq \pi_{nc}$. Therefore, a value $\pi^*_{1,t} < \pi_{nc}$, which follows at least one cooperative choice from firm 1, means that the rival has taken advantage of the cooperative behaviour of firm 1. This implies that the exploited firm will expand its production and therefore move toward the noncooperative level $q_{nc}$. In particular, we suppose that

$$q_{i,t+1} = q_{i,t} + \alpha_i(q_{nc} - q_{i,t}) \quad \text{if} \quad \pi^*_{i,t} < \pi_{nc}, \quad i = 1, 2, \quad 0 < \alpha_i < 1. \tag{$A$}$$

Second, firm 1 knows that it is always $\pi^*_{1,t} + \pi^*_{2,t} \leq 2\pi_c$. Therefore, if $\pi^*_{2,t} > \pi_c$, then it has to be $\pi^*_{1,t} < \pi_c$: again the rival is exploiting the cooperative behaviour of firm 1, and again there will be the incentive for the latter to increase the production level moving toward the non-cooperative level $q_{nc}$. So we suppose that

$$q_{i,t+1} = q_{i,t} + \alpha_i(q_{nc} - q_{i,t}) \quad \text{if} \quad \pi^*_{j,t} > \pi_c, \quad i, j = 1, 2, \quad j \neq i, \quad 0 < \alpha_i < 1. \tag{$B$}$$

Third, we must acknowledge that firms need to be a bit cooperative in their behaviour, otherwise they will end up in the Cournot-Nash (sub-optimal) equilibrium. Therefore, we assumed that one of the cases characterized by a cooperative attitude from firm 1 is when $\pi^*_{1,t} \geq \pi^*_{2,t}$. Here, there is no evidence of recent cheating behaviour from firm 2, and this appears a good incentive for firm 1 to assume a conciliating conduct as well, and therefore reduce the output. Accordingly, we suppose that

$$q_{i,t+1} = q_{i,t} - \beta_i(q_{i,t} - q_c) \quad \text{if} \quad \pi^*_{i,t} \geq \pi^*_{j,t}, \quad i,j = 1,2, \quad j \neq i, \quad 0 < \beta_i < 1. \qquad (C)$$

Finally, we needed another crucial condition imposing a little more altruistic mood in firms' behaviour, in order to be sure of reaching the cooperative equilibrium. This was done by assuming that firms move their output toward the cooperative level also when their average output is lower than the rival's, up to a fraction $\varepsilon_i$ of its own average profit $\pi^*_{i,t}$. In other words, there is some open set where firms behave cooperatively even though they are earning less than the opponent.[9] Therefore, we assume that

$$q_{i,t+1} = q_{i,t} - \beta_i(q_{i,t} - q_c) \quad \text{if} \quad \pi^*_{i,t} < \pi^*_{j,t} \quad \text{and} \quad (\pi^*_{j,t} - \pi^*_{i,t}) \leq \varepsilon_i \pi^*_{i,t}$$
$$i,j = 1,2, \quad j \neq i, \quad 0 < \beta_i < 1. \qquad (D)$$

Rules for firm 2 are defined symmetrically.

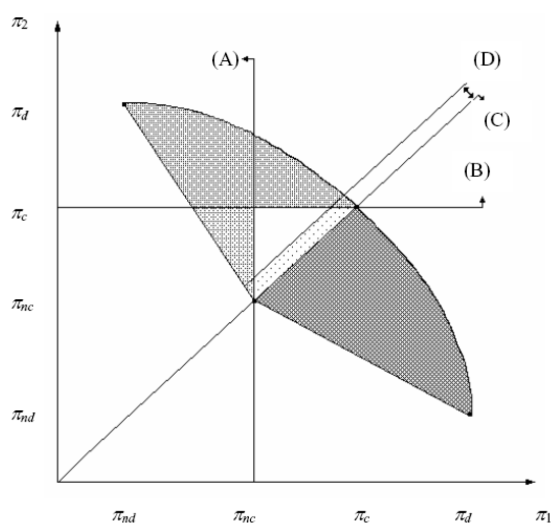



Figure 2: *The profit space and the effects of the "good strategy" rules.*

[9] In his model with two dichotomic choices ("tough" and "easy"), Smale gives reason of this condition by saying that it "...is an expression that we should play a little on the easy side". See Smale (1980), p. 1621.

Figure 2 shows the space of profits for firm 1, together with the areas where our previous conditions hold (shaded accordingly). It is worth to note that in Figure 2 a non–shaded area also appears: here players are not compelled to any rule, and therefore can behave as they wish.
In analogy with Smale, a player's strategy is called good when it satisfies conditions ($A$) through ($D$). Thus, a good strategy requires that each firm should not let itself be exploited, but also that it has to play more on the cooperative side.

# 4 The stability of the 'good' strategy: a proof

In order to demonstrate the convergence of our set of behavioural rules toward the point $(\pi_c, \pi_c)$, we performed some numerical simulations.
We considered two identical firms, whose parameters of interest were as follows:

$$a = 8; \quad b = 1; \quad c_1 = c_2 = c = 0; \quad \alpha_1 = \alpha_2 = 0.01; \quad \beta_1 = \beta_2 = 0.005; \quad \varepsilon_1 = \varepsilon_2 = 0.015\,.$$

With reference to the non-shaded area, we supposed that (with only one exception: see below) firms can randomly choose whether to increase or decrease their production. Besides, we assumed a number of interactions equal to 500.[10] We identified four representative simulations, indicated as (a), (b), (c) and (d), in which the initial choices by both firms were the following:

(a) $q_1 = q_2 = q_{nc}$;

(b) $q_1 = q_2 = q_d$;

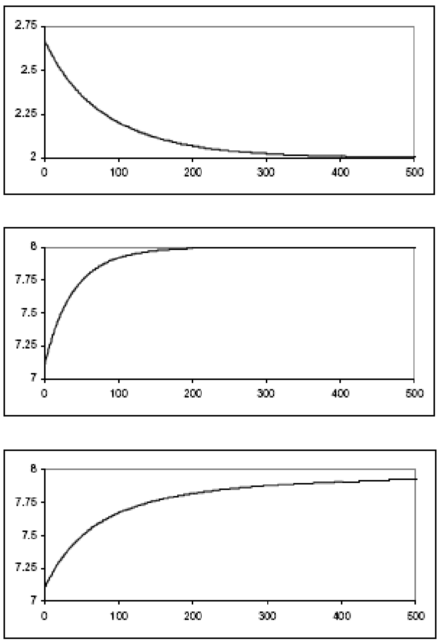
(c) $q_1 = q_{nd}$; $q_2 = q_d$;

(d) $q_1 = q_d$; $q_2 = q_{nd}$; furthermore, in the non-shaded area firms would have always chosen to increase production (assuming a non-cooperative disposition, then).

For each hypothesis, the dynamics of $q_i$, $\pi_i$ and $\pi_i^*$, $i = 1, 2$, are reported in Figures 3 through 6.
Regarding simulations (a) and (b), we observe how the two curves (one for each firm) are always perfectly overlapping (see Figures 3 and 4): hence, firms follow the same behaviour. The only difference is that in the first situation they start from the non–cooperative level of production, while in the second their initial (simultaneous) move is

[10]Given the value of parameters $a$, $b$ and $c$, under our assumptions the various solution pairs are: $q_{nc} = 2.66$ and $\pi_{nc} = 7.11$; $q_c = 2$ and $\pi_c = 8$; $q_d = 3$ and $\pi_d = 9$; $q_{nd} = 2$ and $\pi_{nd} = 6$.

to deviate from a supposed cooperative choice of the rival. In the latter case, in spite of this double cheating, we observe that the good strategy allows the cooperative solution to occur.



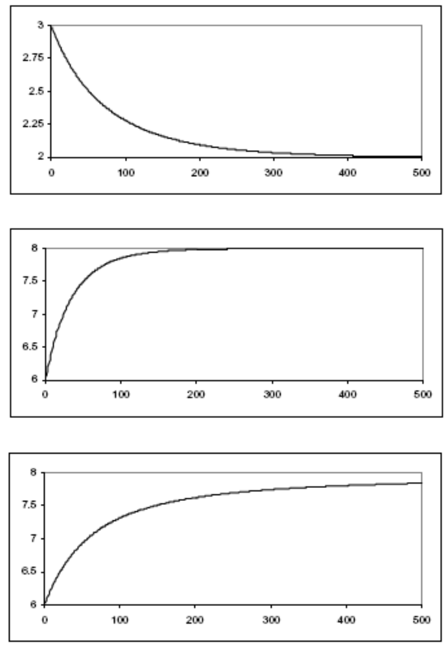
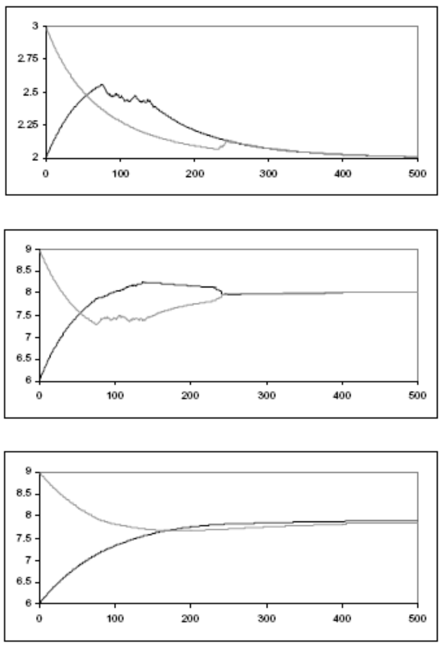




Figure 3: (on the left) *Case (a): dynamics of* $q_i$*,* $\pi_i$ *and* $\pi_i^*$ *(*$i = 1, 2$*), respectively, when both firms start producing* $q_{nc}$*.*

Figure 4: (on the right) *Case (b): dynamics of* $q_i$*,* $\pi_i$ *and* $\pi_i^*$ *(*$i = 1, 2$*), respectively, when both firms start producing* $q_d$*.*

Simulation (c) corresponds to the situation where in the first period one of the firms (firm 1, here characterized by a black line) starts with a cooperative behaviour but has to face (at least initially) a disloyal partner (firm 2, grey line). Again, the dynamics is convergent to the cooperative outcome (see Figure 5). The trembling movement of the curves in the first two graphs is due to the fact that in those stages firms are in the non-shaded area of Figure 2, where they are free to choose whether to increase or decrease current production, and select one of these actions with equal probability. We calculated that in this simulation firm 1 fell in the free-choice area 134 times (94 for firm 2).
Simulation (d) allows to evaluate what happens when in the free-choice area we exogenously impose a non-cooperative attitude (both firms always increase production).

Figure 6 shows that, regardless of this, the proposed set of rules guarantees the achievement of the cooperative point, also if the convergence is slightly slower than before, as firms happen to fall 207 times as a whole in that area (95 for firm 1 and 112 for firm 2).



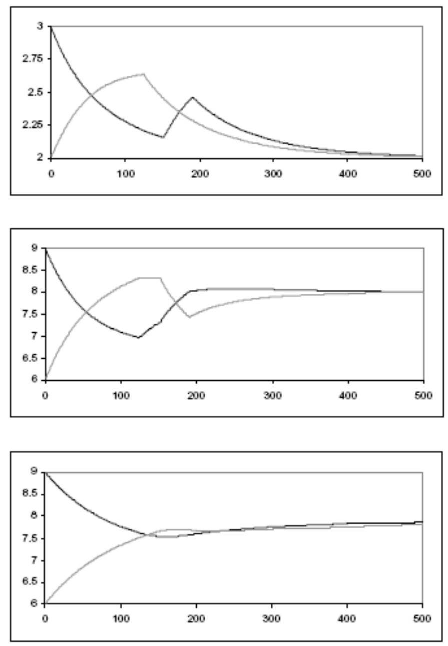




Figure 5: (on the left) *Case (c): dynamics of* $q_i$*,* $\pi_i$ *and* $\pi_i^*$ $(i = 1, 2)$*, respectively, when firm 1 starts producing* $q_{nd} = q_c$ *(black line) and firm 2 starts producing* $q_d$ *(grey line).*

Figure 6: (on the right) *Case (d): dynamics of* $q_i$*,* $\pi_i$ *and* $\pi_i^*$ $(i = 1, 2)$*, respectively, when firm 1 starts producing* $q_d$ *(black line), firm 2 starts producing* $q_{nd} = q_c$ *(grey line), and in the free-choice area both of them always choose to increase production.*

Finally, in order to prove the robustness of our rules in the attainment of the cooperative outcome, we performed a fifth simulation (e), where we randomly chose the initial levels of production: in particular, we fixed $q_1 = 6$ and $q_2 = 0$. Figure 7 shows that the convergence is nonetheless assured, although a higher number of interactions is needed in comparison with the previous cases (actually, we set 2000 interaction here). The dynamics is clear: firm 2 raises its production up to the Cournot value, and leaves it unchanged until firm 1's conciliatory behaviour - deriving from condition $(C)$ - makes possible a cooperative mood, in accordance with condition $(D)$.

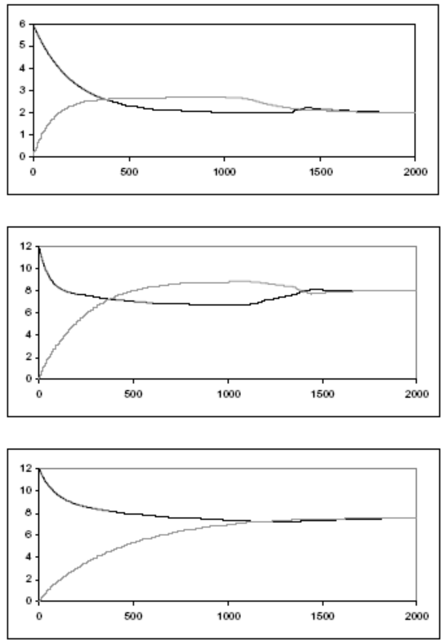


Figure 7: *Case (e): dynamics of* $q_i$*,* $\pi_i$ *and* $\pi_i^*$ *(*$i = 1, 2$*), respectively, when firm* 1 *starts producing* $q_1 = 6$ *(black line) and firm* 2 *starts producing* $q_2 = 0$ *(grey line).*

Even if through a group of simulations, we had reached an unambiguous and encouraging result, summarized in the following proposition.

**Proposition 4.1** *If both players play good strategies* $q_1'$ *and* $q_2'$*, then the solution* $((q_1', q_2'), (\pi_c, \pi_c))$ *is a globally stable solution.*

This means that, given any $q_{i,1}$ (and hence $\pi_{i,1}$), the sequence $\pi_{i,2}, \pi_{i,3}, \ldots$, which has been generated by the dynamics reported in (2.5), converges to the cooperative outcome. Our evidence allowed us to also formulate some other propositions.

**Proposition 4.2** *If firm* 1 *plays a good strategy, then:*

$$\liminf \pi_{1,t}^* \geq \pi_{nc} \, .$$

**Proof.** Actually, for firm 1 the worst situation is when firm 2 always chooses $q_{nc}$. This implies that, if in time 1 it is $q_1 = q_c$, firm 1's profit will be equal to $\pi_{nd}$ (and also

$\pi^*_{1,t} = \pi_{nd}$), but subsequently - according to condition $(A)$ - it will gradually approach $\pi_{nc}$. □

**Proposition 4.3** *If firm 1 plays a good strategy, the best choice for firm 2 is to play a good strategy as well, given that:*

$$\limsup \pi^*_{2,t} \leq \pi_c.$$

**Proof.** In this case, with firm 1 following a good strategy, firm 2's average profit $\pi^*_{2,t}$ will approach $\pi_c$ or $\pi_{nc}$ according to whether it decides to play a good strategy or not. Since $\pi_c > \pi_{nc}$, the optimal strategy for firm 2 is to adopt a good strategy too. □

**Proposition 4.4** *If both firms play good strategies, then for $\varepsilon > 0$ there exists a $t_n$ such that for $t > t_n$ and for any initial $q_1 \in S_q$, it is:*

$$|\pi^*_{i,t} - \pi_{i,t}| < \varepsilon, \quad \textit{all } \pi_{i,t} \in S_\pi \, .$$

**Proof.** Since $\pi^*_{i,t} = \frac{(t-1)\pi^*_{i,t-1}+\pi_t}{t}$, we can substitute this value in $|\pi^*_{i,t} - \pi_{i,t}| < \varepsilon$, obtaining $\left|\frac{(t-1)\pi^*_{i,t-1}+\pi_t}{t} - \pi_{i,t}\right| < \varepsilon$. Rearranging, we get: $\left|\frac{t-1}{t}(\pi^*_{i,t-1} - \pi_t)\right| < \varepsilon$. This is true for a sufficiently large $t$ and for firms playing good strategies, which also ensures that $\pi^*_t$ and $\pi_t$ converge to the same value. □

**Proposition 4.5** *If firms are identical and both play good strategies, then for $\varepsilon > 0$ there exists a $t_n$ such that for $t > t_n$ and for any initial $q_1 \in S_q$ and $q_2 \in S_q$, it is:*

(4.5a) $\quad |(\pi^*_{1,t} + \pi^*_{2,t}) - 2\pi_c| < \varepsilon;$

(4.5b) $\quad |\pi^*_{1,t} - \pi^*_{2,t}| < \varepsilon \, .$

**Proof.** According to Proposition 4.3, the adoption of good strategies drives both firms toward the cooperative outcome $\pi_c$. This yields (4.5a) and (4.5b). □

# 5 Epilogue

Vittorio and I were extremely happy: we had shown that, if firms interact repeatedly, they realize that the non-cooperative solution is Pareto-inferior and can alter their behaviour according to a learning process and a set of rules that guarantee an optimal and stable outcome, regardless of any (explicit or implicit) agreement. After publishing our paper (Cafagna and Coccorese, 2005) we began to think to possible future developments regarding this topic and our scientific collaboration as a whole. Particularly, we aimed to provide an analytical proof of the convergence and stability of our model, and started a series of discussions and conjectures, during which Vittorio again stood out for his intelligence and competency.

However, in that period he was being more and more caught by his latest studies on the theory of sound, on which he was working hard - as usual - with interest and passion. So we met rarely, but he always used to promise me that we would have soon begun to study, discuss and write again.

Two messages bear witness to this intent. One goes back to September 2005, when he was organizing the 2nd Sound and Music Computing Conference, which on his initiative was held in the University of Salerno (November 24-26, 2005): "*There are less than two months to the Conference, and we are in a crucial stage now. Anyway, I hope to meet you soon, at least for a coffee. From December onward I will be a new man, free as a bird and also on sabbatical leave. It would be nice to start working together again*".
He sent me the second message around Christmas 2005: "*The Conference is over: everything went all right. I am a free man now. I am going to Boston for few days. When I come back, I hope to meet you and start our collaboration again*".

It is not difficult to find in these few lines Vittorio's distinguishing marks: enthusiasm, wish of freedom, love for research. Today we can only have memory of this, but even the memory of a man can teach a lot.

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