PhD course on ℓ-groups and polyhedral geometry

Fundamental results in the theory of ℓ-groups.

1. Hölder theorem (1901) "Every Archimedean linearly ordered ℓ-group embeds into $(\mathbb{R}, +, -, 0, \min, \max)$"
2. Weinberg theorem (1965) "$(\mathbb{Z}, +, -, 0, \min, \max)$ generates the variety of ab. ℓ-group"
3. Bernau theorem (1965) "Every Archimedean ℓ-group is commutative"
4. Hölder − linearly ordered = Yosida's representation (1940's)
5. Hölder − Archimedean = Hahn's embedding theorem (1907)
6. Hahn − linearly ordered = Conrad–Harvey–Holland (1962)
7. Yosida + finitely presented = Baker–Beynon (1970's)
8. Yosida + vector lattice + complete in the norm = Kakutani duality (1940's)
9. Mundici's equivalence (1986) = "The category of abelian ℓ-groups with strong order unit is equivalent to MV-algebras"

Def: A group $(G,+,-,0,\leq)$ is said to be *partially ordered* if it $\leq$ is a partial order *compatible with +* (*or translation invariant*), i.e., $\forall x \forall y \forall z \in G \quad x \leq y \Rightarrow x+z \leq y+z$ and $z+x \leq z+y$

A partially ordered group is called *ordered group* (*o-group*) if the order is linear and it is called *lattice (ordered) group* (*l-group*) if the order is a lattice order.

Remark If G is a partially ordered group, define

$G^+ := \{ x \in G \mid x > 0 \}$ *the positive cone of G*

the partial order of G is completely determined by G^+

$$x > y \quad \text{iff} \quad x-y > 0 \quad \text{iff} \quad x-y \in G^+$$

In general if $P \subseteq G$ such that P is closed under addition, (normal), and such that $P \cap -P = \emptyset$ the P induces a partial order that makes G a partially ordered group

$\boxed{g+P-g=P}$

Lemma 1 $+$ distributes over $\vee$ and $\wedge$

Proof $$x+(y\vee z)=(x+y)\vee(x+z)$$

$$\left.\begin{array}{l} y\vee z\geq y \Rightarrow x+(y\vee z)\geq x+y \\ y\vee z\geq z \Rightarrow x+(y\vee z)\geq x+z \end{array}\right\} x+(y\vee z)\geq(x+y)\vee(x+z)$$

Suppose that $w\geq(x+y)\vee(x+z) \Rightarrow \begin{array}{l} w\geq x+y \\ w\geq x+z \end{array} \Rightarrow \begin{array}{l} w-x\geq y \\ w-x\geq z \end{array} \Rightarrow$

$$\Rightarrow w-x\geq y\vee z \Rightarrow w\geq x+(y\vee z)$$

$\square$

Lemma 2
1) $x\geq 0 \Rightarrow -x\leq 0$
2) $x\leq y \Rightarrow -y\leq -x$

Proof
1) $x\geq 0 \Rightarrow x-x\geq -x \Rightarrow 0\geq -x$
2) $x\leq y \Rightarrow x-y\leq 0 \Rightarrow -(x-y)\geq 0 \Rightarrow y-x\geq 0 \Rightarrow -x\geq -y$ $\square$

Corollary
1) $-(x\wedge y)=-x\vee -y$
2) $-(x\vee y)=-x\wedge -y$

Remark By the previous corollary if a partially ordered group is a $\wedge$-semilattice or $\vee$-semilattice, then it is an ℓ-group.

Lemma 3 $x-(x\wedge y)+y = x\vee y$

Proof

$$x-(x\wedge y) = x+(-x\vee -y) = (x-x)\vee(x-y) = (y-y)\vee(x-y)$$
$$= (x\vee y)-y$$
$$\Rightarrow x-(x\wedge y) = (x\vee y)-y \Rightarrow x-(x\wedge y)+y = x\vee y \qquad \square$$

Definition If $g\in G$ $\quad g^+ := g\vee 0 \qquad g^- := (-g)\vee 0$

$$|g| = g^+ + g^-$$

Notice that $g+g^- = g+(-g\vee 0) = (g-g)\vee g+0 = 0\vee g = g^+$

$$\Downarrow$$
$$g = g^+ - g^-$$

$$g^+\wedge g^- = (g+g^-)\wedge g^- = (g+g^-)\wedge(0+g^-) = (g\wedge 0)+g^- = -g^- + g^- = 0$$
$$\Downarrow$$
$g^+\wedge g^- = 0 \qquad g^+, g^-$ are orthogonal.

Lemma 4 $\quad g = |g| \iff g \geq 0$

<u>Proof</u> Recall $|g| := g^+ - g^-$

($\Leftarrow$) Suppose $g \geq 0$ $\quad \overset{g^+}{\overset{\shortparallel}{g \vee 0}} = g$ and $\overset{g^-}{\overset{\shortparallel}{-g \vee 0}} = 0 \implies g = g^+ + g^- = |g|$

($\Rightarrow$) Suppose $g = |g| \implies g^+ - g^- = g^+ + g^- \implies 2g^- = 0 \implies g^- = 0$

$\implies -g \vee 0 = 0 \implies -g \leq 0 \implies g \geq 0$

Lemma 5 Any ℓ-group is torsion-free $\qquad mg = 0 \implies m(-g) = 0$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -mg$

Proof. Suppose $mg := \underbrace{g + \dots + g}_{m \text{ times}} = 0$

$m(g \vee 0) = (g \vee 0) + (m-1)(g \vee 0) = \cancel{mg} \vee (m-1)g \vee \dots \vee g \vee 0 = (m-1)(g \vee 0)$

So $m(g \vee 0) = (m-1)(g \vee 0) \implies g \vee 0 = 0$

Similarly one proves that $-g \vee 0 = 0$. Hence $g = 0$ $\square$