The Optimal Dartboard?

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Our challenge is to determine the best arrangement of the numbers $1, 2, \ldots, 20$ on a dartboard. This problem has been tackled before but we consider a new constraint and a different optimality criterion that lead to an original solution.

1 Problem definition

There are $20! \approx 2 \times 10^{18}$ possible arrangements of the numbers $1, 2, \ldots, 20$ on a dartboard and $19!/2 \approx 6 \times 10^{16}$ distinct cycles that allow for reflection and rotation. Our aim is to determine a cycle that is optimal in some sense. There are three constraints that we wish to impose on any cycle as follows.

1. Penalise mistakes by overambitious players. This was apparent in the standard dartboard designed by Brian Gamlin in 1896, in which large numbers tend to be adjacent to small numbers.
2. Alternate odd and even numbers. This parity criterion was proposed by Eastaway and Haigh [1]. It is particularly appealing because it induces a degree of symmetry and ensures a challenging endgame.
3. Exhibit rotational quasi-symmetry. We propose this criterion to ensure that similar clusters of adjacent sectors all around the dartboard offer similar rewards to players.

The notation that we adopt is as follows. Define the twenty numbers reading clockwise from the top of a dartboard to be x_i for $i = 1, 2, \ldots, 20$. The ordered set $(x_1, x_2, \ldots, x_{20})$ forms a cycle and we define $x_0 = x_{20}$ for convenience. The standard dartboard has the arrangement

$$(20, 1, 18, 4, 13, 6, 10, 15, 2, 17, 3, 19, 7, 16, 8, 11, 14, 9, 12, 5)$$

as illustrated in Figure 1.

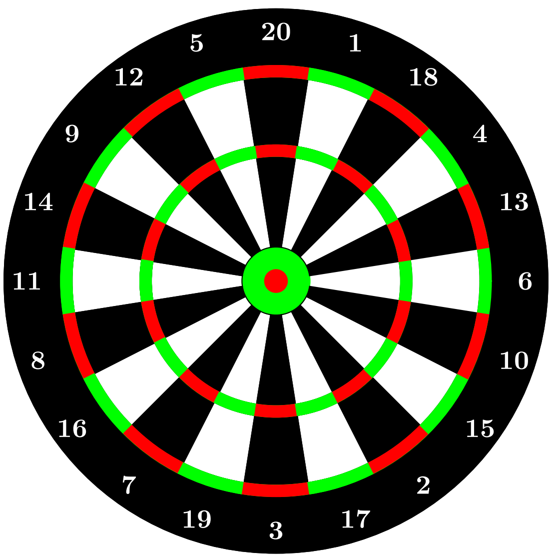

Figure 1: Gamlin's dartboard

2 Aggregate differences

The optimality criteria that dominate published research into this problem involve maximising aggregate penalty measures corresponding to specific forms of the p-norm

$$\|\mathbf{d}\|_p = \left(\sum_{i=1}^{20} |d_i|^p \right)^{1/p} \tag{1}$$

in terms of the differences

$$d_i = x_i - x_{i-1}$$

between pairs of adjacent numbers for $i = 1, 2, \ldots, 20$. In particular, Everson and Bassom [2] and Eastaway and Haigh [1] consider only the Manhattan norm $\|\mathbf{d}\|_1$, Selkirk [3] and Eiselt and Laporte [4] additionally consider the Euclidean norm $\|\mathbf{d}\|_2$, and Cohen and Tonkes [5], Curtis [6] and Liao et al. [7] consider solutions for general p. Articles [3], [2] and [1] use analytical methods of solution as an interesting problem in combinatorics, whereas articles [4], [5], [6] and [7] use computational methods of solution as a form of the NP-hard, travelling salesman problem.

It is easy to prove that $9!10!/2 \approx 7 \times 10^{11}$ distinct cycles achieve $\max \|\mathbf{d}\|_1 = 200$. As noted in [1], these all take the form of alternating numbers from the sets $\{1, 2, \ldots, 10\}$ and $\{11, 12, \ldots, 20\}$, and it is impossible for any of these solutions to satisfy the parity requirement of Constraint 2. In contrast, a unique cycle corresponding to

$$(20, 1, 19, 3, 17, 5, 15, 7, 13, 9, 11, 10, 12, 8, 14, 6, 16, 4, 18, 2) \tag{2}$$

achieves $\max \|\mathbf{d}\|_2 = \sqrt{2,642}$ and we illustrate the corresponding dartboard in Figure 2. Without loss of generality, we orientate the cycle so that $x_1 = 20$ in accordance with the standard dartboard and the number 1 is as close as possible to its standard position (the same sector in this case). This solution also satisfies $\|\mathbf{d}\|_1 = 200$ but fails the parity requirement miserably.

Singmaster [8] suggested that differences between adjacent numbers do not necessarily penalise mistakes by overambitious players, a view with which we concur. For example, the cycle

$$(20, 1, 12, 2, 19, 3, 17, 4, 16, 5, 15, 6, 14, 7, 13, 8, 11, 9, 18, 10)$$

achieves $\max \|\mathbf{d}\|_1 = 200$ but the differences either side of 12 are 11 and 10, whereas the differences either side of 18 are only 9 and 8, so players fare considerably better on average by aiming for the 18 sector rather than the 12 sector. He proposed that sums of adjacent numbers should be about equal instead, this measure does indeed satisfy Constraint 1. To achieve this, he sought to minimise the variance of these sums and proved algebraically the existence of a unique solution that achieves this optimality, which is equivalent to Cycle (2) that maximises the Euclidean norm $\|\mathbf{d}\|_2$. He also proved that this criterion corresponds to minimising the cyclic autocorrelation of lag one, which is an excellent interpretation of the requirement implied by Constraint 1.

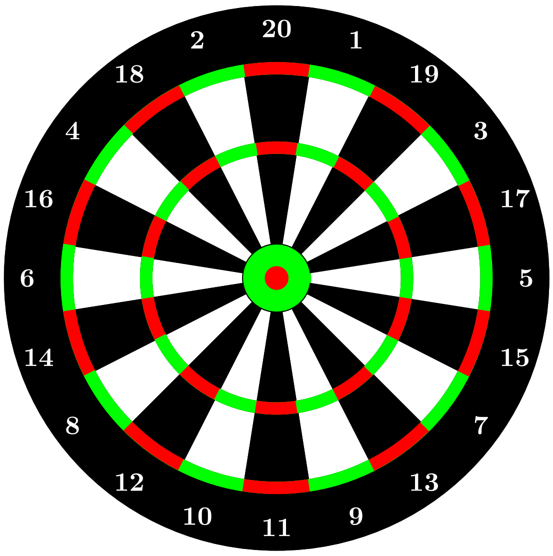

Figure 2: Selkirk's and Singmaster's dartboard

3 Aggregate central sums

Although we support Singmaster's recommendation to consider sums of adjacent numbers rather than differences [8], we recall that the solution based on minimising the variance of these sums fails to satisfy the parity requirement of Constraint 2. Seeking other possible variations on this theme inspired by equation (1), which might lead to solutions that satisfy the parity requirement, we consider optimality criteria that involve minimising aggregate penalty measures corresponding to specific forms of the p-norm

$$\|\mathbf{c}\|_p = \left(\sum_{i=1}^{20} |c_i|^p\right)^{1/p} \tag{3}$$

in terms of the central sums

$$c_i = s_i - \bar{s} \tag{4}$$

of pairs of adjacent numbers, where

$$s_i = x_i + x_{i-1} \tag{5}$$

for $i = 1, 2, \ldots, 20$ and

$$\bar{s} = \frac{1}{20}\sum_{i=1}^{20} s_i = 21 \tag{6}$$

is the arithmetic mean of sums of pairs of adjacent numbers. Note that

$$\|\mathbf{c}\|_2 = \sqrt{\sum_{i=1}^{20} (s_i - \bar{s})^2},$$

which is proportional to the standard deviation of the sums. Consequently, minimising the Euclidean norm $\|\mathbf{c}\|_2$ is equivalent to Singmaster's optimality criterion [8] with a unique solution corresponding to Cycle (2).

Exhaustive enumeration may be used to determine solutions that minimise the Manhattan norm $\|\mathbf{c}\|_1$. We do not pursue this matter further here, though initial calculations suggest that $\min \|\mathbf{c}\|_1 = 18$, which is achieved by Cycle (2) and other cycles including

$$(20, 1, 18, 3, 16, 5, 14, 7, 12, 9, 11, 10, 13, 8, 15, 6, 17, 4, 19, 2).$$

Neither of these solutions satisfies the parity requirement in Constraint 2 and it is possible that none of the solutions that minimise $\|\mathbf{c}\|_1$ does. Our reason for not considering this criterion in more detail is because of the additional rotational quasi-symmetry requirement that we proposed in Constraint 3.

Aggregate measures such as those used for the optimality criteria in equations (1) and (3) do not guarantee rotational quasi-symmetry. To demonstrate this, consider Cycle (2), which optimises the norms $\|\mathbf{d}\|_2$, $\|\mathbf{d}\|_1$, $\|\mathbf{c}\|_2$ and possibly $\|\mathbf{c}\|_1$. The sum of absolute differences in the semicircle that surrounds the number 20 is 149, whereas the sum of absolute differences in the opposite semicircle is 51. Together, these sum to $\max \|\mathbf{d}\|_1 = 200$ but this disparity in the subtotals reveals a clear and undesirable lack of rotational quasi-symmetry. This particular cycle does not exhibit a similar disparity when considering central sums, which are evenly distributed around the board. However, it is conceivable that similar disparities might occur more generally for optimal cycles based on aggregate measures corresponding to the p-norms $\|\mathbf{d}\|_p$ and $\|\mathbf{c}\|_p$.

4 Maximum absolute central sum

To avoid the possibility of violating the symmetry requirement of Constraint 3, while retaining the measure of central sums to satisfy the penalty requirement of Constraint 1 and imposing the parity requirement of Constraint 2, we propose instead to minimise the maximum norm

$$\|\mathbf{c}\|_\infty = \lim_{p\to\infty}\left(\sum_{i=1}^{20} |c_i|^p\right)^{1/p} = \max_i |c_i| \tag{7}$$

subject to the parity constraint

$$\frac{c_i}{2} \in \mathbb{Z} \tag{8}$$

for $i = 1, 2, \ldots, 20$ where c_i is defined by equations (4) to (6). Any solution to this problem will ensure that each large number has small numbers next to it, whereas each medium number has medium numbers next to it. It will also ensure that odd numbers alternate with even numbers around the board and that all clusters of sectors offer similar rewards to the players.

As each number is adjacent to two others, it helps to note that this criterion equivalently sets the 1-2-1 moving averages to be as similar as possible and so is neutral around the board for players that hit the desired sector half the time. Better players (with weighting 1-w-1 for $w > 2$) should aim for larger numbers such as 20 and 19 whereas worse players (with weighting 1-w-1 for $w < 2$) should aim for smaller numbers such as 1 or 2. Several complementary publications address this issue of where best to aim on a dartboard, including Percy [9].

Theorem

The cycle defined by

$$(20, 1, 18, 5, 14, 9, 10, 13, 6, 17, 2, 19, 4, 15, 8, 11, 12, 7, 16, 3) \tag{9}$$

uniquely minimises the maximum norm of equation (7) subject to the parity constraint of Relationship (8). □

Proof

We first note that adjacent sums of pairs must differ because all numbers appear only once:

$$s_i - s_{i(\text{mod}20)+1} = x_{i-1} - x_{i(\text{mod}20)+1} \neq 0$$

for $i = 1, 2, \ldots, 20$. Consequently, not all pairs can sum to 21 and hence $\|\mathbf{c}\|_\infty > 0$. Moreover, all sums of pairs s_i are odd because of the parity constraint $c_i/2 \in \mathbb{Z}$ for $i = 1, 2, \ldots, 20$. Thus, the constrained minimum satisfies $\|\mathbf{c}\|_\infty \geq 2$ and the best that we can hope for is that some pairs sum to 21, while equal numbers of pairs sum to 19 and 23. Graph theory reveals a unique cycle that meets this aspiration with $\|\mathbf{c}\|_\infty = 2$.

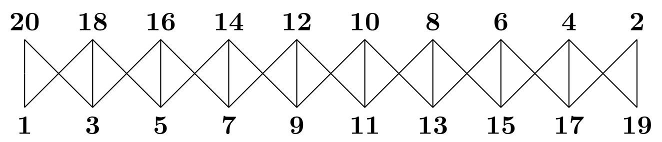

Figure 3: Construction of optimal cycle

Figure 3 shows the set of even numbers above the set of odd numbers with all feasible links that sum to 19, 21 or 23. All numbers on a dartboard are adjacent to two others, so each vertex in Figure 3 must have degree two by virtue of connecting to two edges. This implies that we must remove eight edges from the graph. As 20 already has degree 2, it must connect to 1 and 3. Similarly, 1 already has degree 2 and so must connect to 20 and 18. We now observe that 18 cannot connect with 3 because then the set $\{20, 1, 18, 3\}$ would form a disjoint subgraph, which would violate the constraint that the numbers $1, 2, \ldots, 20$ form a cycle. Consequently, we must remove the edge linking 18 and 3. Proceeding inductively in this way from left to right, it is apparent that all of the internal vertical edges must be removed, leaving a unique solution that corresponds to Cycle (9) and so the proof is complete. ■

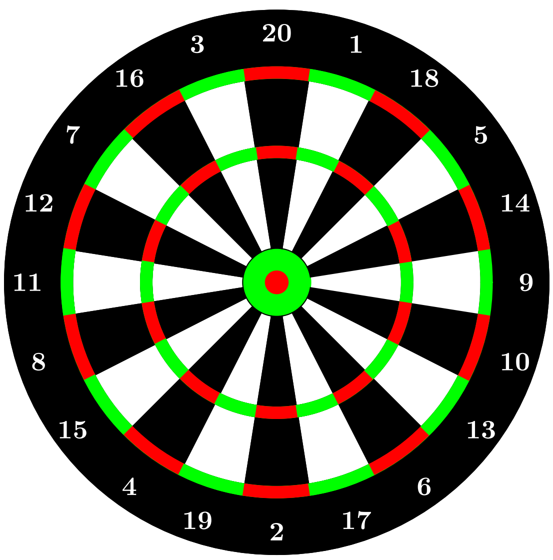

Figure 4: New optimal dartboard

Figure 4 illustrates the new design of dartboard that the preceding theorem generates. Again without loss of generality, we orientate the cycle so that $x_1 = 20$ in accordance with the standard dartboard and the number 1 is as close as possible to its standard position (the same sector in this case). Notice that 8 of the 20 numbers are in the same positions as for the standard dartboard illustrated in Figure 1.

5 Conclusions

In this paper, we determine an optimal arrangement for the numbers on a dartboard. To achieve this, we propose a new constraint relating to rotational quasi-symmetry and a new criterion based on minimising the maximum norm of central sums of pairs of adjacent numbers. We also incorporate a desirable and recently proposed parity constraint to construct a unique cycle displayed in Figure 4, which is optimal according to this new criterion.

This paper argues that criteria other than minimisation of the maximum norm $\|\mathbf{c}\|_\infty$ subject to $c_i/2 \in \mathbb{Z}$ for $i = 1, 2, \ldots, 20$ are flawed due to non-compliance with the three constraints of penalty, parity and symmetry imposed in Section 1. Nevertheless, it is interesting to assess how well the different designs score under all of these criteria. Table 1 presents this information concisely, though it is important to note that the last column contains our preferred measure and only the last row satisfies the parity constraint. Selkirk's and Singmaster's (S & S) dartboard is optimal under all of these criteria but has the disadvantage of grouping all odd numbers together and all even numbers together. Our new design is at least as good as Gamlin's dartboard under all of these criteria and has the added advantage of perfect parity. ❑

Dartboard	max. $\|\mathbf{d}\|_1$	max. $\|\mathbf{d}\|_2$	min. $\|\mathbf{c}\|_1$	min. $\|\mathbf{c}\|_2$	min. $\|\mathbf{c}\|_\infty$
Gamlin's	198	49.8	52	13.5	5
S & S	200	51.4	18	4.2	1
New design	198	50.9	36	8.5	2

Table 1: Comparison of new and existing dartboard arrangements

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